J. DIFFERENTIAL GEOMETRY
5 (1971) 427-435

TWO CLASSES OF CLASSICAL SUBGROUPS OF Diff(M)

J. LESLIE

Introduction

Sometime ago in a letter J. Eells asked us whether it was possible to give a differential structure to the automorphisms of a G-structure similar to the one for the group of diffeomorphisms. At this time the author does not know whether it is possible to give a local modelling of the group of automorphisms $D_G(M)$ of an arbitrary G-structure on a compact manifold M, although many of the formal properties of a manifold are satisfied for $D_G(M)$. The purpose of this note is to give a manifold structure to $D_G(M)$ in two cases:

(i) when the Lie algebra of G is closed under matrix multiplication, and (ii) it contains the case when G is elliptic in the sense of Spencer [11].

The author takes this occasion to thank D. C. Spencer for his encouragement during the preparation of this note.

1. Analysis in topological vector spaces

All topological vector spaces appearing in this paper are Hausdorff complete locally convex topological vector spaces over the real numbers R; continuous functions will be called C^0 functions when convenient.

Definition 1. Let $U \subset E$, $V \subset F$ be open sets in topological vector spaces E and F, and suppose that G is a third topological vector space. A function $f : U \times V \to G$ is n times differentiable at $(\xi, \eta) \in U \times V$ in the first (resp. second) variable, if f is $n-1$ times differentiable in the first (resp. second) variable at (ξ, η), and there exists a continuous symmetric n-multilinear function

$$(\partial^n f/\partial x^n)(\xi, \eta) : \underbrace{E \times \cdots \times E}_{n\text{-times}} \to G$$

$$(\text{resp. } (\partial^n f/\partial y^n)(\xi, \eta) : \underbrace{F \times \cdots \times F}_{n\text{-times}} \to G)$$

such that

Received April 30, 1970 and, in revised form, September 21, 1970. Supported in part by National Science Foundation Grant GP-7952X1.

$$F(v) = f(\xi + v, \eta) - f(\xi, \eta) - (\partial f/\partial x)(\xi, \eta)(v) - \cdots - (1/n!)(\partial^n f/\partial x^n)(\xi, \eta)(v, \cdots, v)$$

$$(\text{resp. } G(v) = f(\xi, \eta + v) - (\partial f/\partial y)(\xi, \eta)(v) - \cdots - (1/n!)(\partial^n f/\partial y^n)(\xi, \eta)(v, \cdots, v))$$

has the property that

$$\varphi(t, v) = F(tv)/t^n, \quad t \neq 0,$$
$$= 0, \quad t = 0$$
$$(\text{resp. } \gamma(t, v) = G(tv)/t^n, t \neq 0, \gamma(t, v) = 0, t = 0)$$

is continuous on $R \times E$ (resp. $R \times F$) at $(0, v)$, $v \in E$ (resp. $v \in F$).

Throughout this paper when we speak of derivative and differentiability it will be with respect to the above definition. Setting $F = \{0\}$ we find the definition of an n times differentiable function $f : U \to G$. It is obvious how to generalize the above definition to any number of variables.

Definition. f is said to be C^n in the first (resp. 2nd) variable if f is n times differentiable at each $(x, y) \in U \times V$, and

$$\partial^m f/\partial x^m \ (\text{resp. } \partial^m f/\partial y^m)$$

defines a continuous function

$$U \times V \times E \times \cdots \times E \to G$$
$$(\text{resp. } U \times V \times F \times \cdots \times F \to G) \qquad \text{for } 0 \leq m \leq n .$$

The following four propositions are easy to prove, but useful to state.

Proposition 1. *Let E and F be Banach spaces, and U an open subset of E. If $f: U \to F$ is C^n in the above sense, then f is C^{n-1} in the Fréchet sense.*

Proof. Note that C^0 in the above sense and C^0 in the Fréchet sense are the same, namely, continuous. By definition C^1 in the above sense implies C^0 in the Fréchet sense. Suppose it has been established that C^k in the above sense implies C^{k-1} in the Fréchet sense for $k < n$, and suppose $f: U \to F$ is C^n in the above sense. As $Df: U \times E \times \cdots \times E \to F$ is continuous at $x_0 \in U$, it follows that there exist an open neighborhood U_0 of x_0 in U and a positive constant K so that $|D^n f(U_0)| < K$, where $D^n f(x)$ in $L_s^n(E, F)$ is the map induced by fixing x from $D^n f$. Thus for y in U_0 we have

$$|D^{n-1}f(y)(\alpha_1, \cdots, \alpha_n) - D^{n-1}f(x_0)(\alpha_1, \cdots, \alpha_n)|$$
$$< \int_0^1 |D^n f(x_0 + t(y - x_0), (y - x_0), \alpha_1, \cdots, \alpha_{n-1}) dt|$$
$$< \int_0^1 |D^n f(x_0 + t(y - x_0), (y - x_0), \alpha_1, \cdots, \alpha_{n-1})| dt$$
$$< K|y - x_0||\alpha_1| \cdots |\alpha_{n-1}| .$$

Proposition 2. *Let E and G be complete locally convex topological vector spaces, and suppose that F is a closed subspace of G and that U is an open subset of E. A function $f : U \to F$ is C^n if and only if $i \circ f$ is C^n, where $i : F \to G$ is the canonical injection.*

Our proof makes use of the following

Lemma. *Let E and F be topological vector spaces, and suppose $U \subset E$ be an open convex subset. If $f : U \to F$ is C^r, and $D^r f : U \times E \times \cdots \times E \to F$ is C^s in the first variable, then f is C^{r+s}.*

Proof. Let $A_{s,r}(x, \alpha_1, \cdots, \alpha_s, \beta_1, \cdots, \beta_r)$ be the symmetrization of

$$\frac{(r+s)!}{(s+1)!r!}(\partial^s/\partial x^s)D^n f(x, \alpha_1, \cdots, \alpha_s; \beta_1, \cdots, \beta_r) .$$

Now

$$0 = f(x+th) - f(x) - Df(x,th) - \cdots - (1/(r-1)!)D^{r-1}f(x, th, \cdots, th) - \frac{1}{r!}\int_0^1 D^r f(x + \tau th, th, \cdots, th)d\tau$$

$$= f(x+th) - f(x) - Df(x,th) - \cdots - (1/(r-1)!)D^{r-1}f(x, th, \cdots, th)$$
$$- \frac{1}{r!}\int_0^1 [D^r f(x, th, \cdots, th) + (\partial/\partial x)D^r f(x, th, \cdots, th, t\tau h)$$
$$+ \cdots + \frac{1}{(s-1)!}(\partial^{s-1}/\partial x^{s-1})D^r f(x, th, \cdots, th, t\tau h, \cdots, t\tau h)$$
$$+ \frac{1}{s!}\int_0^1 (\partial^s/\partial x^s)D^r f(x + \sigma\tau th, th, \cdots, th, t\tau h, \cdots, t\tau h)d\sigma]dt ,$$

$$\tau = f(x+th) - f(x) - Df(x,th) - \cdots - \frac{1}{(r-1)!}D^{r-1}(fx, th, \cdots, th)$$
$$- \frac{1}{r!2}(\partial/\partial x)D^r f(x, th, \cdots, th) - \cdots$$
$$- \frac{1}{r!s!}(\partial^{s-1}/\partial x^{s-1})D^r f(x, th, \cdots, th)$$
$$- \frac{1}{r!s!}\int_0^1\int_0^1 (\partial^s/\partial x^s)D^r f(x + \sigma\tau th, th, \cdots, th, t\tau h, \cdots, t\tau h)d\sigma d\tau .$$

If we subtract the last expression divided by t^{r+s} from

$$\{f(x+th) - f(x) - A_{0,1}(x, th) - \cdots - \frac{1}{r!}A_{0,r}(x, th, \cdots, th)$$
$$- \frac{1}{(r+1)!}A_{1,r}(x, th, \cdots, th) - \cdots - \frac{1}{(r+s)!}A_{s,r}(x, th, \cdots, th)\}/t^{r+s} ,$$

we obtain

$$\int_0^1\int_0^1\Big[\frac{1}{(sh)!r!}(\partial^s/\partial x^s)D^r f(x, th, \cdots, th)$$
$$- \frac{1}{s!r!}(\partial^s/\partial x^s)D^r f(x + \sigma\tau th, th, \cdots, th, t\tau h, \cdots, t\tau h)\Big]d\sigma d\tau / t^{r+s}$$
$$= \int_0^1\int_0^1\Big[\frac{1}{(s+1)!r!}(\partial^s/\partial x^s)D^r f(x, h, \cdots, h)$$
$$- \frac{1}{s!r!}(\partial^s/\partial x^s)D^r f(x + \sigma\tau th, h, \cdots, h, th, \cdots, th)\Big]d\sigma d\tau$$
$$= \phi(t, x, h)\ .$$

ϕ is a continuous function so that $\phi(0, x, h) = 0$.

Corollary. *$f : U \to F$ is C^{r+s} if and only if f is C^r and $D^r f(x, \alpha_1, \cdots, \alpha_r)$ is C^s in the first variable.*

Definition. Let $\{B_i\}_{i\geq 0}$ be a sequence of Banach spaces so that

(i) B_{i+1} is a subspace of B_i for the underlying vector space structure,

(ii) the injection $k_i^{i+1} : B_{i+1} \to B_i$ induces a continuous function $\{B_i\}_{0\leq i\leq\infty}$ is called a Banach chain where $B_\infty = \bigcap_{i\geq 0} B_i$ is considered to have the inverse limit topology.

Proposition 3. *Let $\{B_i^1\}$ and $\{B_i^2\}$ be Banach chains. Suppose $U \subset B_\infty^1$ is an open set, and $f : U \to B_\infty^2$ is a function so that for every positive integer r there exist a strictly increasing sequence of positive integers k, a monotonically increasing positive integral valued function $\alpha_r(k)$, and a collection of open sets $U_{k,r} \subset B_k^1$ so that $U \subset U_{k,r} \cap B^1$. If f extends to a C^r function $f_{k,r} : U_{k,r} \to B_{\alpha(k)}^2$, then f is a C^∞ function.*

The proof of the above proposition follows from the definitions as does

Proposition 4. *Let E and G be complete locally convex topological vector spaces and $U \subset E$ be open, and suppose F is a closed subspace of G. Then f is C^n if and only if $i \circ f : U \to G$ is C^n, where $i : F \to G$ is the canonical injection.*

2. The automorphisms of two classes of G-structures

We recall that Diff(M), $D(M)$, $D_n(M)$, and $\mathscr{D}_n(M)$ are respectively the group of diffeomorphisms with the C^∞ topology, the connected component of the indentity in Diff(M), the group of C^n diffeomorphisms of M, and the vector space of C^{n-1} right invariant vactor fields on $D_n(M)$, and [5, p. 267] that the tangent space at $f \in$ Diff(M) can be represented by $\mathscr{D}_f(M) = \{\alpha : M \to TM \mid \tau\circ\alpha = f\}$. An admissible chart at $f \in$ Diff(M) can be given as follows: From [5], $\exists t > 0$ such that setting

$$S_t = \{\alpha \in \mathscr{D}_f(M) \,|\, |\alpha|_1 < t \quad \text{where} \quad |\alpha|_1 \text{ is the } C^1 \text{ norm}\}$$

and defining $e(\alpha)(x) = \exp_r(\alpha(x))$ where $\exp_r$ is the Riemannian exponential we obtain a chart at f. Multiplication and inversion defiine smooth maps.

It is now useful to put some properties of the classical subgroups of Diff (M) in our terminology. Let $C^\infty(E)$ be the space of sections of a locally trivial fiber bundle $\pi : E \to M$, where M is compact.

Proposition 1. *$C^\infty(E)$ can be given the structure of a smooth C^∞ manifold in such a way that the tangent space of $C^\infty(E)$ at $s \in C^\infty(E)$ can be represented by the nuclear space of smooth sections of $TF(E) \xrightarrow{\pi \circ p} M$ with the C^∞ topology.*

Using Palais' notion of a bundle spray (see [9]) this proposition can be proved by the same methods used in [5] to show that Diff(M) admits a smooth manifold structure. Hereafter when $C^\infty(E)$ is considered as a manifold it will be with respect to this structure.

Proposition 2. *Suppose $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$, where M is a smooth compact manifold, are smooth fiber bundles. If $f : E_1 \to E_2$ is a bundle homomorphism over M, then $f_* : C^\infty(E_1) \to C^\infty(E_2)$ is a smooth function.*

The following is immediate from the definitions.

Proposition 3. *Let $E = M \times M \xrightarrow{\pi_1} M$ be projection on the first factor, and $J_r \xrightarrow{\alpha_r} M$ be the fiber space of r jets with projection on the source. Then the jet extension map $j_r : C^\infty(E) \to \gamma_r(M) = C^\infty(J_r)$ is smooth.*

Designate by μ_r the fiber space of invertible r-jets of smooth endomorphisms of M. μ_r is an open submanifold of J_r so that $\alpha_r \,|\, \mu_r$ is a principal fibration.

Definition. Let $\pi : E \to M$ be an arbitrary smooth locally trivial fibration. A Lie differential operator of order r on Diff (M) is a function $D = f_* \circ j_r$, where $j_r :$ Diff $(M) \to \gamma_r$ is the canonical map, and $f : \gamma_r \to E_2$ is a smooth morphisms of fiber bundles over M; so that

(i) $D^{-1}(D(e)) = G$ is a subgroup of Diff (M),

(ii) $D(gh) = D(h)$ for $g \in G$.

When $D :$ Diff $(M) \to C^\infty(E)$ is a Lie differential operator, $D^{-1}(D(e))$ is called a classical subgroup of Diff (M). Note that a Lie differential operator defines a smooth function $D :$ Diff $(M) \to C^\infty(E)$.

Proposition 4. *Let $D^{-1}(D(e)) = G$ be a classical subgroup of* Diff (M), *suppose* exp : $\mathscr{D}(M) \to$ Diff (M) *is the Lie exponential, and let $g = \{g \in \mathscr{D}(M) \,|\, T_h D(R_h(g)) = 0$ for all $h \in$* Diff (M), *where R_h is induced by right multiplication by $h\}$. Then* exp $(tX) \in G$ *for every t if and only if $X \in g$.*

Proof. Suppose exp $(tX) \in G$ for all t. Then

$$T_h D(R_h X) = \left(\frac{d}{dt}\right)_{t=0} D(\exp(tX)h) = 0\,,$$

since D is constant on Gh.

Now for $X \in g$ set $f(t) = D(\exp(tX))$, so that

$$f'(t) = T_{\exp(tX)}D(X(\exp(tX)) = T_{\exp(tX)}D(R_{\exp(tX)}X) = 0 .$$

Thus $f(t) = f(0) = D(e)$ and $f(t) \in G$.

Proposition 5. *Under the hypotheses of Proposition 9, g is a Lie subalgebra of $\mathscr{D}(M)$.*

Proof. Since g is obviously a vector space, we need only to show that g is closed under the bracket operation for vector fields. Let $X, Y \in g$, and suppose φ_t generates X (i.e., $T_{t=0}\varphi_t = X$) and ϕ_t generates Y. Then we have

$$\begin{aligned} T_hD(R_h[X, Y]) &= T_hD\left(\lim_{t\to 0} \frac{1}{t}(R_hX - R_h(ad\, \phi_t X))\right) \\ &= \lim_{t\to 0} \frac{1}{t}\{T_{t=0}(D(\phi_t \circ h)) - T_{t=0}(D(\phi_t\phi_t^{-1}h))\} \\ &= \lim_{t\to 0} \frac{1}{t}\{T_{t=0}D(h) - T_{t=0}D(h)\} \\ &= 0 . \end{aligned}$$

Hence $[X, Y] \in g$. q.e.d.

g is called the Lie algebra of G.

A subgroup of the full linear group $GL(n)$ will be called locally convex when it is locally convex for the canonical vector space structure on $M(n)$.

Proposition 6. *Let G be a Lie subgroup of $GL(n)$. If its Lie algebra $g \subset M(n)$ is closed under matrix multiplication, then G is a locally convex open subset of $I + g$.*

Theorem. *Let M be a compact smooth manifold, without boundary, of dimension n, and let G be a subgroup of $GL(n)$ whose Lie algebra is closed under matrix multiplication. Suppose the group of the tangent bundle of M can be reduced to G (i.e., M admits a G-structure). Then the automorphisms of the G-structure $D_G(M)$ admits a manifold structure locally diffeomorphic to its tangent space at a point, and $f : U \to D_G(M)$ is smooth if and only if $i \circ f : U \to \mathrm{Diff}(M)$ is smooth, where $i : D_G(M) \to \mathrm{Diff}(M)$ is the canonical homomorphism.*

Proof. Choose a G connection on M, and let $\exp_G : TM \to M$ be the exponential map associated with this G-structure. $\widetilde{\exp}_G : \mathscr{D}(M) \to F(M, M)$ given by $\widetilde{\exp}_G(\alpha)(x) = \widetilde{\exp}_G \circ \alpha(x)$ is such that there exists a real number $t > 0$ so that $\widetilde{\exp}_G | S_t(0)$ is a diffeomorphism onto an open neighborhood of the identity in $\mathrm{Diff}(M)$.

Cover M by normal coordinate neighborhoods $\{U_i\}$ with respect to the given G-connection, and consider X in the Lie algebra $\mathscr{G}$ of $D_G(M)$. We shall prove that $\widetilde{\exp}_G(X) \in D_G(M)$ for $|X|_1 < t$. Locally with respect to the normal coordinate $\widetilde{\exp}_G(X)(x)$ can be written as $x + X_x$ for t sufficiently small. Suppose $g_t = \exp_l(tX)$. Then we have

$$D_x \widetilde{\exp}_G(X)(x, \alpha) = D \exp_G \circ D_x D_{t=0} g_t(x, \alpha)$$
$$= D \exp_G \circ D_{t=0} D_x g_t(x, \alpha) = D \exp \circ g(x, \alpha)$$

where $g(x, \alpha) = (x, \alpha, X_x, \gamma(\alpha))$, γ being in the Lie algebra of G. Thus

$$D_x (\widetilde{\exp}_G(X))(x, \alpha) = (x + X_x, \alpha + g_x(\alpha)) ,$$

where g_x is in the Lie algebra of G. For X sufficiently C^1 small, g_x is small and thus $\alpha \to \alpha + g_x(\alpha) \in G$.

Now suppose $(D_x g)(x, \alpha) = (g(x), h(\alpha))$ where $h \in G$ and $g \in \operatorname{Diff}(M)$; $h \in GL(n)$ is given by the connection on M. Now $\widetilde{\exp}_G^{-1}(g)(x) = (x, g(x) - x)$. Consider $h_t(x) = x + t(g(x) - x)$ so that $D_x h_t(x, \alpha) = (h_t(x), \alpha + t\gamma(\alpha))$ where γ is in the Lie algebra of G. Thus $D_x h_t(x, \alpha) = (h_t(x), g(\alpha))$ where $g \in G$, and $H(t) = h_t$ is a smooth arc in $\operatorname{Diff}(M)$ so that $H(-1, 1) \subset D_G(M)$. Hence $D_{t=0}H = \{x \to (x, g(x)) - x\} \in T_e D_G(M)$.

Similarly, $\exp_G : \{X \cdot g \mid g \in D_G(M),\ X \in g,\ \text{and}\ |X|_1 < t\} \to D_G(M)$ maps diffeomorphically onto a neighborhood of g. By the same procedure as in [5] one obtains that $D_G(M)$ is a manifold where multiplication defines a smooth function and $g \to g^{-1}$ is smooth.

The final statement of the theorem follows from Proposition 2, § 1.

Corollary (*see* [12]). *The automorphisms of a multifoliate structure on a compact manifold satisfy the conclusions of the above theorem.*

Definition 1. A chain of Hilbert spaces $\{H_i\}_{0<i<\infty}$ is a chain of Banach spaces where the H_i are Hilbertable spaces.

It is classical that a nuclear space can be given as the H_∞ in a chain of Hilbert spaces.

In the category of chains of Hilbert spaces as in the category of chains of Banach spaces (see [6]), a mapping $f : U \to H^2_\infty$, $U \subset H^1_\infty$ being open, is said to be C^r when there exists a sequence of integers $k \to \infty$ such that f extends to C^r mappings $f_k : U_k \to H^2_{\lambda(k)}$ where $U_k \subset H^1_k$ is open and $U = H^1_\infty \cap U_k$. Proposition 3 of § 1 states that C^r in the category of Banach or Hilbert chains is a stronger notion than C^r in the category of nuclear spaces in terms of Definition 1, § 1.

We shall now review the Ebin-Omori notion of inverse limit Hilbert manifolds as applied to the group of diffeomorphisms.

Definition 2. A sequence of C^∞ Hilbert manifolds $\{X_r\}$ is called an inverse limit Hilbert system (or an I.L.H. system) when

(i) $X_{r+1} \subset X_r$,

(ii) there is a Hilbert chain $\{H_r\}$ such that for $x \in X_\infty$ there exist charts at x:

$$\varphi_r : U_r \to X\ , \qquad U_r \subset H_r \text{ being open.}$$

An I.L.H. system $\{X_r\}$ is called an inverse limit Hilbert system of groups (or an I.L.H.G. system) when X_{r+1} is a subgroup of X_r and multiplication and inversion define smooth maps in the category of Banach chains.

Now let M be a compact smooth (C^∞) manifold, and $\pi : E \to M$ be a Riemannian vector bundle over M. For an integer $s \geq 0$, let $H^s(E)$ be the completion of $j_s(C^\infty(E))$ in the norm involving the integral of the inner product in $J^s(E)$, and set $C^k(E)$ equal to the space of sections of E of class C^k. Then by the Sobolev theorems one has canonically

$$H^{n/2+k+1}(E) \subset C^k(E) \subset H^k(E)\ ,$$

where $n = \dim(M)$. Similarly, when M and N are manifolds and $s > n/2 + 1$, it makes sense to talk of an H^s map from M to N by looking at the mapping locally. So let $H^s(M, N)$ be the space of H^s maps from M to N for $s > \dim M/2 + 1$, and set $D^s(M) = \{H^s(M, M) \cap D_1(M)\}$ for $S > n/2 + 1$.

By the same construction as on p. 433 one may show that $D^s(M)$ is a smooth Hilbert manifold modelled on $H^s(TM)$. Since $D_\infty(M)$ is an inverse limit Banach group (see [10]), it follows from the Sobolev theorems that $D_\infty(M)$ is an inverse limit Hilbert group.

In [8] Omori proved

Theorem 2. *Let M be a compact manifold, and $D : C^\infty(TM) \to C^\infty(E)$ be a linear differential operator of order l. Then there exists a vector bundle over $D^s(M)$, $\varepsilon^s \to D^s(M)$ with fiber at $g \in D^s(M) H^s(E) \circ g$ so that D defines a vector bundle morphism*

$$\begin{array}{ccc} TD^{s+l} & \xrightarrow{\tilde{D}} & s \\ \downarrow{\scriptstyle \pi_{s+l}} & & \downarrow{\scriptstyle \pi_s} \\ D^{s+l} & \xrightarrow{i_s^{s+l}} & D^s \end{array}$$

with $\tilde{D}(\alpha \circ g) = D(\alpha) \circ g$, where $\alpha \in H^{s+l}(TM)$ and $g \in D^{s+l}(M)$.

Definition. A linear differential operator of order $l, D : C^\infty(E_1) \to C^\infty(E_2)$, is called closed when D extends to maps $D^s : H^{s+l}(E_1) \to H^s(E_2)$ with closed range.

Theorem 3. *Let G be a classical subgroup of* Diff (M). *If its Lie algebra g is the kernel of a closed linear differential operator $d : C^\infty(TM) \to C^\infty(E)$, then G contains a closed normal subgroup H_∞ and* $\exp(g) \in H_\infty$, *where $\{H_i\}$ is a* sub-I.L.H.G. *of $\{D^s(M)\}$ with H_s a closed submanifold of $\{D^s(M)\}$.*

Proof. Let K_s be the complement of $d_{s+l}(H^{s+l}(TM))$ in $H^s(E)$. By means of Theorem 2 and Proposition 6 [4, p. 45] we obtain that $\operatorname{Ker}(\tilde{d}_{s+l}) = \operatorname{Ker}(\tilde{d}_{s+l} \oplus \mathrm{id}_K) : U \times H^{s+l}(TM \oplus K \to U \times H^s(E))$ is a closed sub-bundle of $TD^{s+l}(M)$, so there exists a connected subgroup H_{s+l}, which is also a C^∞

manifold, with tangent space at the identity $= g_{s+l}$ = the closure of g in $H^{s+l}(TM)$ (see [6]). From the construction it hence follows that $g = \bigcap_s g_s$.

Now let $D : \mathrm{Diff}\,(M) \to C^\infty(\xi)$ be the non-linear differential operator of order k which defines $G = D^{-1}(D(e))$. Then D extends to smooth $D_{s+k} : D^{s+k}(M) \to H^s(\xi)$ (see [9, p. 67]). It is easy to see that $H_{s+k} \subset D^{-1}_{s+k}(D(e))$ and that the arc component of $D^{-1}_{s+k}(D(e)) \subset H_{s+k}$; thus H_{s+k} is normal in $D^{-1}_{s+k}(D(e))$. H_{s+k} is locally closed in $D^{s+k}(M)$ and hence closed being a topological subgroup of $D^{s+k}(M)$.

Remark 1. When $TD_{s+k} : TD^{s+k}(M) \to TH^s(\xi)$ are surjective, the D_{s+k} are submersions, $D^{-1}_{s+k}(D(e))$ are submanifolds of $D^{s+k}(M)$, and G itself may be regarded as an I.L.H.G. In this case a mapping $f : U \to G$, U being open in some vector space, is C^n if and only if $f : U \to H_s$ is C^n for all s.

Remark 2. The differential structures of H_∞ and G are locally the same in some sense due to the fact that if U is a convex open set of a topological vector space, and $f : U \to G$ is continuous with $x_0 \in U$, then $f(U) \cdot f(x_0)^{-1} \in H_\infty$.

Bibliography

[1] R. Abraham, *Foundations of mechanics,* Benjamin, New York, 1967.
[2] C. Chevalley, *Theory of Lie groups,* Princeton University Press, Princeton, 1946.
[3] D. Ebin, *The manifold of Riemannian metrics,* Bull. Amer. Math. Soc. **72** (1968) 1001–1003.
[4] S. Lang, *Introduction to differentiable manifolds,* Interscience, New York, 1962.
[5] J. Leslie, *On a differential structure for the group of diffeomorphisms,* Topology **6** (1967) 263–271.
[6] ——, *Some Frobenius theorems in global analysis, J.* Differential Geometry **2** (1969) 279–297.
[7] D. Mongomery & L. Zippin, *Topological transformation groups,* Interscience, New York, 1965.
[8] H. Omori, *On I.L.H. properties of mapping spaces,* to appear.
[9] R. Palais, *Foundations of global non-linear analysis,* Benjamin, New York, 1968.
[10] S. Smale, *Lectures on differential topology,* Notes by R. Abraham, Columbia University, 1962.
[11] D. C. Spencer, *Deformation of structures.* III, Ann. of Math. **18** (1965) 389–450.
[12] ——, *Multifoliate structures,* Ann. of Math. **74** (1961) 52–100.

INSTITUTE FOR ADVANCED STUDY
NORTHEASTERN UNIVERSITY